A New Algorithm for Generating Coefficients of Taylor's Series Expansion of Nonlinear Components with Time Harmonic Excitation

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***Abstract*-Nonlinear components and Taylor's series expansion play key role in parametric amplification and harmonic generation and their analysis. The algorithm presented in this paper is for calculating the coefficients of Taylor's series expansion of nonlinear components such as varactors. The coefficients resulted from the algorithm indicate a good agreement with those from Fourier transform. By relaxing assumptions of small-harmonic terms compared with the fundamental one, for small-signal cases, the algorithm can reach much more accurate results than previous theoretical analysis, also give valuable prospective/anticipation for more interesting practical cases of relatively strong nonlinearity (i.e., a large capacitance ratio) and large amplitude.**

I. INTRODUCTION

In recent years, metamaterials with simultaneously negative permittivity and permeability, also known as left-handed (LH) materials, have inspired new types of components and applications[1]-[4] enabled by their unique electrodynamic properties first predicted by Veselago in 1968 [5]. Specifically, transmission line metamaterials have already led to a wealth of microwave applications based on their linear structures [6]. It is of significant interest when combined with nonlinearity, since it enables frequency tuning, switching between opaque and transparent states, and efficient second-harmonic reflection [7]. Nonlinear left-handed transmission lines (NL LHTL) have been studied for potential device applications [8], [9], and more recently for their insight into the physics of left-handed metamaterials [10]. They have been shown to exhibit mismatched harmonic generation [11], parametric generation and amplification [12], and envelope solitons [9], [13].

Nonlinear components such as varactors and Schottky-barrier diodes have proven very useful in parametric amplification [14] and harmonic generation at microwave frequencies [15], [16]. And Taylor's series expansion play key role in their analysis. A simple small-signal treatment of second harmonic generation (SHG) by waves propagating on a Right-handed nonlinear transmission line (RH NLTL) with a capacitance nonlinearity have been reported in 1986 [15]. The simple theory developed in [15] is then applied to an analytical expression for the amplitude of the third harmonic generated in discrete LH NLTLs [17], [18].

To simplify the analysis, these two previous works on nonlinear PRH and PLH transmission line made some assumptions such as harmonic voltage terms are small compared with the fundamental one, so that higher order terms in the Taylor's series expansion of the charge on varactors can be systematically ignored, where the result can be justified only for small input signals. For the most interesting practical cases of strong nonlinearity (i.e., a large capacitance ratio) and large amplitude, the theoretical analysis is not valid.

The algorithm presented in this paper demonstrates for the first time and in a rigorous manner the generation of the coefficients of Taylor's series expansion of the charge on a varactor for time harmonic excitation. Here the nonlinear voltage-charge relationship of a varactor is taken just as a model. In fact, the algorithm is justified for all nonlinear system with time harmonic excitation.

II. ALGORITHM DESCRIPTION

Consider a varactor which possesses a nonlinear voltage-charge relationship that can be expanded in a convergent Taylor's series

$$q(v) = q(V_0) + q'(V_0)(v - V_0) + \frac{1}{2!}q''(V_0)(v - V_0)^2 + \frac{1}{3!}q'''(V_0)(v - V_0)^3 + \cdots\cdots \quad (1)$$

Where q is the stored charge on the varactor, and V_0 is the dc bias voltage. If ω is the lowest frequency sinusoidal component of v, v and q can be expanded in complex Fourier series as follows:

$$v(z,t) = V_0 + \sum_{n=1}^{\infty}\{V_n(z)e^{jnwt} + V_n^*(z)e^{-jnwt}\} \quad (2)$$

and

$$q(z,t) = Q_0(z) + \sum_{n=1}^{\infty}\{Q_n(z)e^{jnwt} + Q_n^*(z)e^{-jnwt}\} \quad (3)$$

From (2), we can set

$$v - V_0 = \sum_{n=1}^{\infty}\{V_n(z)e^{jnwt} + V_n^*(z)e^{-jnwt}\}$$

$$= A_0 + \sum_{n=1}^{\infty}\{A_n(z)e^{jnwt} + A_n^*(z)e^{-jnwt}\} \tag{4}$$

$$(v - V_0)^2 = \sum_{n=1}^{\infty}\{A_n(z)e^{jnwt} + A_n^*(z)e^{-jnwt}\}$$

$$\sum_{n=1}^{\infty}\{V_n(z)e^{jnwt} + V_n^*(z)e^{-jnwt}\}$$

$$= B_0 + \sum_{n=1}^{\infty}\{B_n(z)e^{jnwt} + B_n^*(z)e^{-jnwt}\} \tag{5}$$

$$(v - V_0)^3 = \sum_{n=1}^{\infty}\{B_n(z)e^{jnwt} + B_n^*(z)e^{-jnwt}\}$$

$$\sum_{n=1}^{\infty}\{V_n(z)e^{jnwt} + V_n^*(z)e^{-jnwt}\}$$

$$= C_0 + \sum_{n=1}^{\infty}\{C_n(z)e^{jnwt} + C_n^*(z)e^{-jnwt}\} \tag{6}$$

……

So then, we can get the coefficients, Q_0 to Q_n, as follows:

$$Q_0 = q(V_0) + q'(V_0)A_0 + \frac{1}{2!}q''(V_0)B_0 + \frac{1}{3!}q'''(V_0)C_0 + \cdots\cdots$$

$$Q_1 = q'(V_0)A_1 + \frac{1}{2!}q''(V_0)B_1 + \frac{1}{3!}q'''(V_0)C_1 + \cdots\cdots$$

$$Q_2 = q'(V_0)A_2 + \frac{1}{2!}q''(V_0)B_2 + \frac{1}{3!}q'''(V_0)C_2 + \cdots\cdots$$

……

$$Q_n = q'(V_0)A_n + \frac{1}{2!}q''(V_0)B_n + \frac{1}{3!}q'''(V_0)C_n + \cdots\cdots$$

The number of the terms for each Q_n can be as many as we want.

The essential point of this algorithm is how to calculate the coefficients $A_0, A_n, \ldots A_n^*$, B_0, $B_n, \ldots B_n^*$, $C_0, C_n, \ldots C_n^*$ and so on.

Here, A_0 is zero, $A_n \ldots A_n^*$ are equal to V_n and V_n^* respectively which are known from input signals. According to the algorithm, B_0, $B_n \ldots B_n^*$ can be obtained from $A_0, A_n \ldots A_n^*$ and $V_0, V_n \ldots V_n^*$, $C_0, C_n \ldots C_n^*$ can be produced from B_0, $B_n \ldots B_n^*$ and $V_0, V_n \ldots V_n^*$, and so on. We found that the patterns from one group of coefficient to another are exactly same. This is the essential to accomplish automatic calculation to get all coefficients from Q_0 to Q_n by programming.

III. Calculation Results and Discussions

Compare the Results of Taylor Series Algorithm to Fourier Transform

Take a varactor as an example, the relationship of the charge and the voltage across the varactor is

$$C(v) = C_{j0}(1 - \frac{v}{\phi})^{-\gamma}$$

$$q(v) = -\frac{C_{j0}\phi}{1-\gamma}(1 - \frac{v}{\phi})^{1-\gamma}$$

So the different order derivations of $q(v)$ can be derived accordingly. Assume a time varying voltage with fundamental frequency ω, $v(t) = \sin(\omega t)$, is fed across the varactor. The amplitudes of different order terms Q_n resulted from the algorithm are shown in Fig.1. In Fig.1b, the result from Fourier transform is also displayed. Compared the two results, it can be found that the algorithm from Taylor series expansion is equivalently valid for evaluating the amplitudes of the terms Q_n.

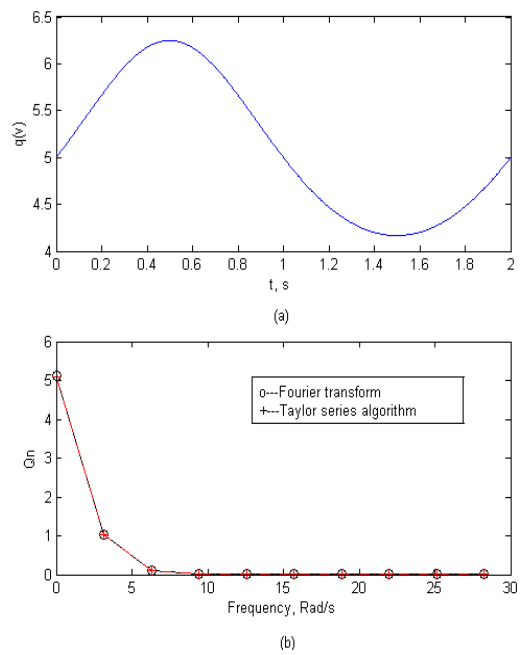

Fig.1 Time series of charge on the varactor (a) and the amplitude of coefficient Q (b) when input v(t)=sin(ωt), C_{j0}=1, Φ=5, γ=2

Fig. 2 is the case when the amplitude of the voltage across the varactor is four times higher than that in Fig.1. It is seen that the results from the presented algorithm also agrees that from the Fourier transform very well. Furthermore, because, during the whole calculation, such assumption as small signals is never made and higher order terms in the Taylor's series

expansion are not necessarily systematically ignored, all the factors which contribute to a certain order term can be involved in calculation, the result is much more accurate than that by the simple theoretical analysis in previous works. In another word, the elements which brought errors in previous simple theoretical analysis can be eliminated in calculation by this algorithm. This performance indicates the algorithm, relaxing the assumption of small signal and small higher harmonic voltage terms compared with the fundamental one, can be anticipated for application to more interesting practical cases, such as relatively strong nonlinearity and large amplitude,

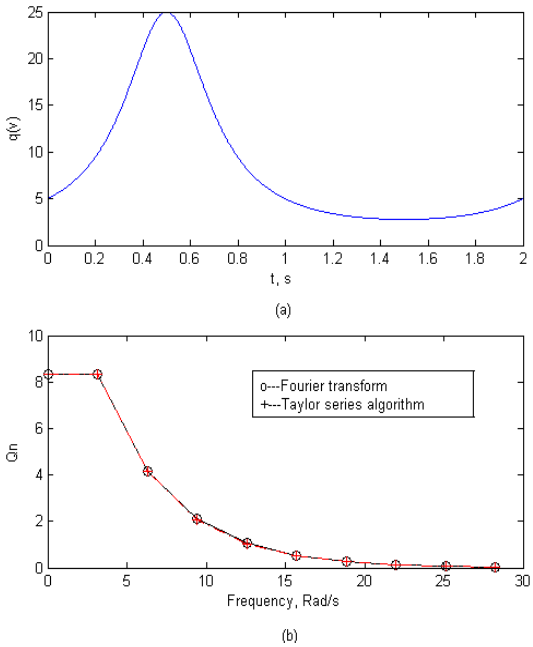

Fig.2 Time series of charge on the varactor (a) and the amplitude of coefficient Q (b)
when input $v(t)=4\sin(\omega t)$, $C_{j0}=1$, $\Phi=5$, $\gamma=2$

For the situation shown in Fig.3 the voltage across the varactor has three frequency components, where the algorithm from Taylor series still works as well as Fourier transform does.

In previous theoretical analysis, even though the voltage across the varactor is assumed with multi frequencies, in order to make the analysis continue, all other frequencies have to be ignored except the fundamental one and the concerned one based on the small signal assumption when calculating the terms Q_n. For example, to get the third order Q_3, only the contributions from the voltages with fundamental frequency and the third harmonic frequency are considered, those from the second harmonic frequency are consequently excluded. So the accuracy of results are much less than that resulted from the algorithm presented in this paper.

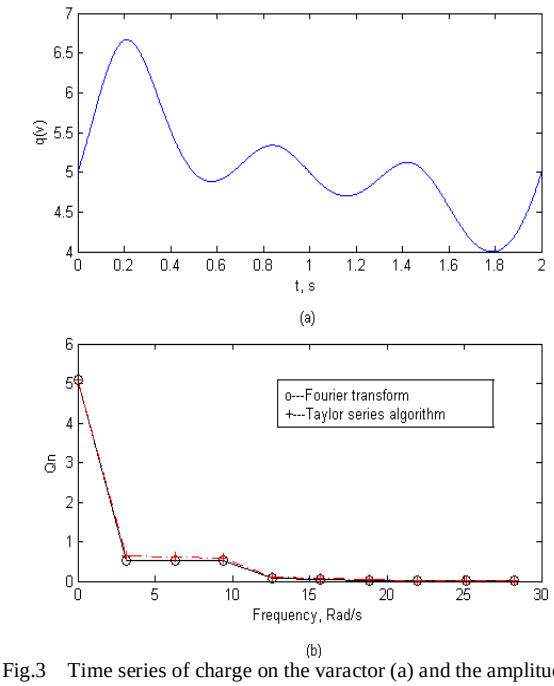

Fig.3 Time series of charge on the varactor (a) and the amplitude of coefficient Q (b)
when input $v(t)=0.5\sin(\omega t)+0.5\sin(2\omega t)+0.5\sin(3\omega t)$, $C_{j0}=1$, $\Phi=5$, $\gamma=2$

IV. Conclusion

This paper demonstrated a algorithm developed for calculating the coefficients of Taylor's series expansion of nonlinear function with time harmonic excitation. This algorithm provides a straight method to get the coefficients related to each of input frequency components by programming, where assumptions such as small signal and small-harmonic voltage terms can be excluded and more accurate calculation can be reached. It also offers a direct way to predict, given input with certain frequency components, how many harmonic terms would be produced in circuits according to the prefered terms of Taylor series.

Compared with the results from Fourier transform of time harmonics q(t) represented by the red line, it can be found that the algorithm from Taylor series expansion, to some extent, is equivalently valid for evaluating the amplitudes of the terms Q, not only for one frequency input, but for multi frequency input as well.

Furthermore, this algorithm relaxes the assumption of small signal and small higher harmonic voltage terms compared with the fundamental one. Therefore it can be applied to interesting practical cases, such as strong nonlinearity and large amplitude. Because higher order terms in the Taylor's series expansion are not necessarily systematically ignored, all the factors which contribute to a certain order term can be involved in calculation, the result could be expected much more accurate than that provided by the simple analytical theory previously used. In addition, it reaches the insight of

harmonic generation and can predict the number of harmonic terms when given a certain excitation.

The algorithm also accomplishes the change from manual calculation to automatic calculation because of the same pattern in which all coefficients can be produced.

REFERENCES

[1] M. Anioniades and G. V. Eleftheriades, "Compact linear lead/lag metamaterial phase shifters for broadband application," IEEE Antennas Wireless Propag. Lett., vol. 2, No. 7, pp. 103-106, 2003.

[2] C. Caloz, A. Sanada, and T. Itoh, "A novel composite right-/left-handed coupled-line directional coupler with arbitrary coupling level and broad bandwidth," IEEE Trans. Microw. Theory Tech., vol. 52, No. 3, pp. 980-992, Mar. 2004.

[3] L. Liu, C. Caloz and T. Itoh, "Dominant mode leaky-wave antenna with backfire-to-endfire scanning capability," Electron. Lett., vol.38, No. 23, pp. 1414-1416, Nov. 2002.

[4] A. Grbic and G. V. Eleftheriades, "Experimental verification of back-wave radiation from a negative refractive index metamaterial," J. Appl. Phys., vol. 92, No. 10, pp. 5930-5935, Nov. 15, 2002.

[5] V. G. Veselego, "The Electrodynamics of substances with simultaneously negative values of permittivity and permeability ," Sov. Phys. Usp., vol.10, no. 4, pp. 509–514, Jan.–Feb. 1968.

[6] C. Caloz and T. Itoh, Electromagnetic metamaterials, transmission line theory and microwave applications, Wiley & IEEE Press, 2005.

[7] I. V. Shadrivov and Yu. S. Kivshar, in Physics of Negative Refraction and Negative Index Materials, Springer Series in Materials Science Vol. 98, edited by C. M. Krowne and Y. Zhang _Springer, New York, 2007

[8] A. A. Zharov, I. V. Shadrivov and Y. S. Kivshar, "Nonlinear properties of left-handed metamaterials," Phy. Review Lett., vol. 91, No. 3, pp. 037401-1-4, Jul. 2003.

[9] I. V. Shadrivov, A. A. Sukhorukov and Y. S. Kivshar, "Nonlinear surface waves in left-handed materials," Phy. Review E 69, pp. 016617-1-9, 2004.

[10] Gupta S. and Caloz C, "Dark and Bright Solitons in Left-Handed Nonlinear Transmission Line Metamaterials," Microwave Symposium, 2007 IEEE/MTT-S International, Vol. Issue, 3-8 June, pp. 979-982, 2007..

[11] A. B. Kozyrev, H. Kim, A. Karbassi, and D. W. van der Weide, Appl. Phys. Lett. 87, 121109, 2005.

[12] A. Kozyrev, H. Kim, and D. van der Weide, Appl. Phys. Lett. 88, 264101, 2006.

[13] A. B. Kozyrev and D. W. van der Weide, Appl. Phys. Lett. 91, 254111, 2007.

[14] R. Landauer, "Parametric Amplification along Nonlinear Transmission Lines," Journal of Applied Physics, vol.31, No.3, pp. 479-484, Mar. 1960.

[15] K. S. Champlin and D. R. Singh, "Small-signal second-harmonic generation by a nonlinear transmission line," IEEE Trans. Microw. Theory Tech., vol. MTT-34, No. 3, pp. 351-353, Mar. 1986.

[16] J. W. Archer, "Millimeter wavelength frequency multipliers," IEEE Trans. Microw. Theory Tech., vol. MTT-29, No. 6, pp. 552-557, June 1981.

[17] A. B. Kozyrev and D. van der Weide, "Nonlinear transmission lines in left-handed media," IEEE MTT-S Digest, pp. 317-320, 2004.

[18] A. B. Kozyrev and D. van der Weide, "Nonlinear wave propagation phenomena in left-handed transmission-line media," IEEE Trans. Microw. Theory Tech., vol. 53, No. 1, pp. 238-245, Jan. 2005.